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ANALYTIC HYPOELLIPTICITY OF CERTAIN SECOND-ORDER EVOLUTION EQUATIONS WITH DOUBLE CHARACTERISTICS

BY

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ABSTRACT. The present article establishes the analytic hypoellipticity (Definition 1.2) of a class of abstract evolution equations of order two, with double characteristics, under the hypothesis that the coefficients are analytic (in a suitable sense; see §2). The noteworthy feature of the main result (Theorem 4.1) is that analytic hypoellipticity holds whenever hypoellipticity does, even when one of the asymptotic eigenvalues $c^j(A)$ fails to be elliptic of order one.

Introduction. The present paper establishes the analytic hypoellipticity (Definition 1.2) of a class of (abstract) evolution equations of order two, with double characteristics—precisely the same class as that studied in the work [1], but under the hypothesis that the coefficients are analytic (in a suitable sense; see §2). We adapt the concatenation method of [1] and use the method of Grushin [2] to derive analytic hypoellipticity from suitable a priori estimates. The noteworthy feature of the main result (Theorem 4.1) is that analytic hypoellipticity holds whenever hypoellipticity does (under the analyticity hypothesis concerning the coefficients)—*even when one of the asymptotic eigenvalues* $c^j(A)$ *fails to be elliptic of order one* (in the latter case our results are essentially particular cases of those of Grushin). Since the equations studied here are microlocal models for the pseudodifferential equations of the kind studied in [4], our main result strongly suggests that, when their total symbols are analytic, the operators II studied in [4] are hypoelliptic analytic if and only if all their asymptotic eigenvalues are.

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1. **Analytic hypoellipticity in the abstract set-up.** Let A be a linear operator, densely defined in a Hilbert space H. We shall assume that A is *unbounded* but it is *selfadjoint, positive-definite* and that it has a *bounded* inverse A^{-1}.

Let J be a given open subset of the real line. We denote by $\mathcal{Q}_A(J)$ the ring of the series in the nonnegative powers of A^{-1} with coefficients in $C^\infty(J)$ which converge in $L(H, H)$ (the Banach space of bounded linear operators of H into H)

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as well as each one of their derivatives in t, uniformly with respect to t on compact subsets of J.

$\mathcal{Q}_A((J))$ will denote the formal power series in A^{-1}, with coefficients in $C^\infty(J)$.

Now we want to state precisely what we mean for analytic hypoellipticity of an operator of the kind

$$P = \sum_{J+k=m} a_{J,k}(t, A)A^J\partial_t^k \tag{1.1}$$

where $a_{J,k}(t, A) \in \mathcal{Q}_A(J)$. Then we have to introduce a scale of "Sobolev spaces" H^s, $s \in \mathbf{R}$, defined by A:

> If $s \geq 0$, H^s is the space of elements $u \in H$ such that $A^s u \in H$, equipped with the norm $\|u\|_s = \|A^s u\|_0$, where $\| \ \|_0$ denotes the norm in H.
>
> If $s < 0$, H^s is the completion for the norm $\|u\|_s = \|A^s u\|_0$.

The inner product in H^s will be denoted by $(\ , \ )_s$.

Whatever s, $m \in \mathbf{R}$, A^m is an isomorphism (for the Hilbert space structure) of H^s onto H^{s-m}.

We denote by H^∞ the intersection of the spaces H^s and by $H^{-\infty}$ their union, the former equipped with the projective limit topology and the latter with the inductive limit topology.

H^∞ and $H^{-\infty}$ with their topology are the strong dual of each other as H^s and H^{-s} are.

We denote by $C^\infty(J, H^\infty)$ the space of the C^∞ functions in J valued in H^∞, equipped with its natural C^∞ topology. It is the intersection of the spaces $C^h(J, H^k)$ as h and k tend to ∞, of the h-continuously differentiable functions defined in J and valued in the Hilbert space H^k.

If K is any compact subset of J, we denote by $C_c^\infty(K, H^\infty)$ the subspaces of $C^\infty(J, H^\infty)$ consisting of the functions which vanish identically outside of K. this is a closed linear subspace of $C^\infty(J, H^\infty)$, then a Fréchet space and we denote by $C_c^\infty(J, H^\infty)$ the inductive limit of the $C_c^\infty(K, H^\infty)$ as K ranges over the compact subsets of J.

Now we can define the space $\mathcal{D}'(J, H^{-\infty})$ of the distributions in J valued in $H^{-\infty}$ as the dual of $C_c^\infty(J, H^\infty)$.

Finally we denote by $A(J, H^\omega)$, $[C^\infty(J, H^\omega)]$, the subspace of $C^\infty(J, H^\infty)$ given by the set of the functions $u(t) \in C^\infty(J, H^\infty)$ such that for every point $t_0 \in J$, there is an open neighborhood J' of t_0 relatively compact contained in J and a constant $C > 0$ such that for every $\alpha, \beta \in \mathbf{N}$

$$\sup_{t\in J'} \|\partial_t^\alpha A^\beta u(t)\|_0 \leqslant C^{\alpha+\beta+1}(\alpha+\beta)! \tag{1.2}$$
$$\left[\sup_{t\in J'} \|A^\beta u(t)\|_0 \leqslant C^{\beta+1}\beta\right]!$$

DEFINITION 1.1. *Let P be as in* (1.1), *P is said to be hypoelliptic in J if given any open subset J' of J and any distribution $u(t) \in \mathcal{D}'(J', H^{-\infty})$,*

$$Pu \in C^\infty(J', H^\infty) \Rightarrow u \in C^\infty(J', H^\infty).$$

We say that **P** *is hypoelliptic at a point* t_0 of J if there is a neighborhood J' of t_0 in J such that **P** is hypoelliptic in J'.

DEFINITION 1.2. **P** *is said to be analytic hypoelliptic in J if* **P** *is hypoelliptic in J and if, given any open subset J' of J and any distribution $u(t) \in \mathcal{D}'(J', H^{-\infty})$,*

$$Pu \in A(J', H^\omega) \Rightarrow u \in A(J', H^\omega).$$

Similarly **P** is *analytic hypoelliptic at a point* t_0 of J if there is a neighborhood J' of t_0 in J such that **P** is analytic hypoellyptic in J'.

We introduce, now, a new scale of Hilbert spaces.

If $s \in \mathbf{R}$ and $s \geqslant 0$, we denote by E_s the set of all elements $u \in H$ for which there is a $v \in H$ such that $u = e^{-sA}v$ and we put in E_s the norm $\|u\|_{E_s} = \|v\|_0$.

If $s \in \mathbf{R}$ and $s < 0$, E_s is the completion of H with the norm $\|u\|_{E_s} = \|e^{sA}u\|_0$.

For every real s, E_s is a Hilbert space with the norm $\| \ \|_{E_s}$ and the following properties are true

(1.3)

(i) For every s and s' such that $s > s'$, E_s is canonically imbedded in $E_{s'}$, with a norm $\leqslant 1$.

(ii) For every s and s' such that $s > s'$, A defines a continuous linear map $A\colon E_s \longrightarrow E_{s'}$, with norm $\leqslant C/(s-s)'$, where C is a positive constant independent on s and s'.

For every real s, we denote by $A_t(J, E_s)$ the space of the analytic functions on J with value in E_s.

REMARK 1.1. It is easy to see that

$$A(J, H^\omega) = \bigcup_{s>0} A_t(J, E_s). \tag{1.4}$$

2. **"Classes" of cut off functions.** Let J denote an open interval centered in the origin; we are going to study the analytic hypoellipticity for "formal" operators of the form

(2.1) $$P = (\partial_t - a(t, A)A)(\partial_t - b(t, A)A) - c(t, A)A = XY - c(t, A)A$$

where

(2.1)′ $$X = \partial_t - a(t, A)A; \qquad Y = \partial_t - b(t, A)A$$

and $a(t, A), b(t, A), c(t, A) \in \mathcal{Q}_A((J))$ which satisfies the following properties:

If $a_0(t)$ and $b_0(t)$ are the leading coefficients of $a(t, A)$ and $b(t, A)$, then

(2.2) $$a_0(0) = b_0(0) = 0$$

(2.3) $$\mathcal{Re}\, a_0'(0) > 0, \qquad \mathcal{Re}\, b_0'(0) < 0.$$

Furthermore, if we write $a(t, A) = \Sigma_k a_k(t)A^{-k}$, $b(t, A) = \Sigma_k b(t)A^{-k}$, $c(t, A) = \Sigma_k c_k(t)A^{-k}$, we suppose that there is a neighborhood U of 0 in C, such that $U \cap \mathbf{R} = J$, on which the C^∞ functions $a_k(t), b_k(t), c_k(t)$ can be extended as holomorphic functions satisfying the inequality

(2.4) $$\{|a_k(z)|, |b_k(z)|, |c_k(z)|\} \leqslant c^{k+1}k!$$

for every integer $k \geqslant 0$ and for every $z \in U$.

Therefore, by Cauchy's inequality, after a shrinking of U, we can suppose that there is a constant $M > 0$ such that

(2.5) $$|\partial_t^\alpha a_k(z)|, |\partial_t^\alpha b_k(z)|, |\partial_t^\alpha c_k(z)| \leqslant c^{k+1}M^\alpha \alpha! k!$$

for every integer $\alpha, k \geqslant 0$ and for every $z \in U$.

Because A is a selfadjoint operator, using the spectral resolution of A, we have that

(2.6) $$A = \int_{-\infty}^{+\infty} \lambda\, dE_\lambda = \int_{\sigma(A)} \lambda\, dE_\lambda, \qquad A^{-1} = \int_{-\infty}^{+\infty} \lambda^{-1}\, dE_\lambda = \int_{\sigma(A)} \lambda^{-1}\, dE_\lambda$$

where $\sigma(A)$ is the spectrum of A.

Below $\{\varphi_k(\lambda)\}$, $k \in \mathbf{N}$, denotes a sequence of continuous real valued functions defined on $\mathbf{R}$.

DEFINITION 2.1. *Let c_1 be a number > 0. We will say that the sequence $\{\varphi_k(\lambda)\}$ belongs to the "class of cut off functions" $[c_1]$ if*

(1) $\varphi_0(\lambda) = 1$ *for every* $\lambda \in \mathbf{R}$,

(2) $0 \leqslant \varphi_k(\lambda) \leqslant 1$ *for every* $k \geqslant 1$ *and* $\lambda \in \mathbf{R}$ *and*

(2.7) $$\varphi_k(\lambda) = \begin{cases} 1 & \forall \lambda \geqslant c_1 k + 1, \\ 0 & \forall \lambda \leqslant c_1 k. \end{cases}$$

Then for every nonnegative integer k, the operator

$$\varphi_k(A) = \int \varphi_k(\lambda)\, dE_\lambda \tag{2.8}$$

is a bounded selfadjoint linear operator defined on H into H.

We define formally

$$\mathbf{P}(\{\varphi_k\}) = (\partial_t - a(t, A, \{\varphi_k\})A)(\partial_t - b(t, A, \{\varphi_k\})A) - c(t, A, \{\varphi_k\})A \tag{2.9}$$

where

$$a(t, A, \varphi_k) = \sum_k a_k(t)\varphi_k(A)A^{-k},$$

$$b(t, A, \varphi_k) = \sum_k b_k(t)\varphi_k(A)A^{-k},$$

$$c(t, A, \varphi_k) = \sum_k c_k(t)\varphi_k(A)A^{-k}.$$

REMARK 2.1. If $c_1 > c$, where c is given by (2.4), for every $\{\varphi_k\} \in [c_1]$ $\mathbf{P}(\{\varphi_k\})$ is a "true" operator, that is the coefficients $a(t, A, \{\varphi_k\})$, $b(t, A, \{\varphi_k\})$ and $c(t, A, \{\varphi_k\})$ are C^∞ functions defined on J with value in $L(H, H)$.

Indeed, take an integer α and let $\| \ \|$ denote the norm $L(H, H)$; we have, by (2.5),

$$\sup_{t\in J} \left\| \sum_0^h (\partial_t^\alpha a_k)(t)\varphi_k(A)^{-k} \right\|$$

$$\leq \sum_{k=0}^h \sup_J |\partial_t^\alpha a_k(t)| \cdot \|\varphi_k(A)A^{-k}\|$$

$$\leq \sum_{k=0}^h c^{k+1}M^\alpha \alpha! k! \cdot (c_1 k)^{-k} \leq cM^\alpha \alpha! \left(\sum_0^\infty \left(\frac{c}{c_1}\right)^k\right),$$

which proves the result.

Then there is a constant M_1 such that for every $\alpha \in \mathbf{N}$

$$\sup_J \{\|\partial_t^\alpha a(t, A, \{\varphi_k\})\|, \|\partial_t^\alpha b(t, A, \{\varphi_k\})\|, \|\partial_t^\alpha c(t, A, \{\varphi_k\})\|\} = M_1^{\alpha+1}\alpha! \tag{2.10}$$

Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are operators of the kind

$$\mathbf{P}_1 = \sum_{J+k=2} F_{J,k}(t)A^J\partial_t^k, \qquad \mathbf{P}_2 = \sum_{J+k=2} G_{J,k}(t)A^J\partial_t^k$$

where $F_{J,k}(t)$, $G_{J,k}(t)$ are analytic functions on J with value in $L(H, H)$ which can be extended as holomorphic functions on the complex neighborhood U of 0 ($J = U \cap \mathbf{R}$) with value in $L(H, H)$ and $F_{0,2}(t) = G_{0,2}(t) = I_H$, I_H being the identity mapping of H.

THEOREM 2.1. *Let us suppose that*

$$\mathbf{P}_1 = \mathbf{P}_2 + \mathbf{R} \tag{2.11}$$

where $\mathbf{R}$ *is a linear operator which has the following property: there is an* $\epsilon > 0$ *such that for every real* s

$$R: \mathcal{A}_t(J, E_s) \longrightarrow \mathcal{A}_t(J, E_{s+\epsilon}) \tag{2.12}$$

is a linear operator of $\mathcal{A}_t(J, E_s)$ *into* $\mathcal{A}_t(J, E_{s+\epsilon})$.

Then if $\mathbf{P}_2$ *is analytic hypoelliptic at* $t = 0$, *the same is true of* $\mathbf{P}_1$.

PROOF. For the proof we will use some results of [3], using the scale of Hilbert spaces E_s.

Let u be an element of $\mathcal{D}'(J, H^{-\infty})$ such that $\mathbf{P}_1 u = f$ where $f \in \mathcal{A}(J, H^{\omega})$, then $f \in \mathcal{A}_t(J, E_0)$, recalling that $E_0 = H$, then if we shrink enough the complex neighborhood U of 0 we can extend f to a holomorphic function on U with values in E_0, i.e. $f \in \mathcal{A}_t(U, E_0)$.

Now we reduce the second order differential equation in t, $\mathbf{P}_1 u = f$, to a first order linear system

$$LU = I\partial_t U - M(t, A)U = F(t)$$

where I is the identity matrix 2×2, $M(t, A)$ a 2×2 matrix and $F(t)$ a two-vector, holomorphically dependent on t varying in U.

Now for (i) and (ii) of (1.3) the hypothesis of Theorem 2.1 in [3] are verified and by the uniqueness of the solution and by Remark 9.3 in [3], for every $s < 0$, u is the restriction of a holomorphic function $\widetilde{u}(t)$ on U (after a shrinking of U) with value in E_s, in particular we have $u \in \mathcal{A}_t(J, E_s)$.

Then by property (2.12), if we choose $s = -\epsilon/2$, we have that $Ru \in \mathcal{A}_t(J, E_{\epsilon/2})$, then by Remark 1.1 we have that $Ru \in \mathcal{A}(J, H^{\omega})$.

Therefore

$$\mathbf{P}_2 u = \mathbf{P}_1 u - Ru \in \mathcal{A}(J, H^{\omega}),$$

then by the analytic hypoellipticity, at $t = 0$, of $\mathbf{P}_2$ we get that also $\mathbf{P}_1$ is analytic hypoelliptic at $t = 0$.

We have now the following consequence

THEOREM 2.2. *Suppose that* c_1 *and* c_2 *are two constants such that*

$$c_1 > c, \qquad c_2 > c, \tag{2.13}$$

where c *is the constant given in* (2.4), *if* $\{\varphi_k\} \in [c_1]$ *and* $\{\psi_k\} \in [c_2]$, *then* $\mathbf{P}(\{\varphi_k\})$ *is analytic hypoelliptic at* $t = 0$ *if and only if this is so for* $\mathbf{P}(\{\psi_k\})$.

PROOF. We have

$$\begin{aligned}
\mathbf{P}(\{\varphi_k\}) &= \mathbf{P}(\{\psi_k\} - \{\psi_k - \varphi_k\}) \\
&= [\partial_t - a(t, A, \{\psi_k\})A + a(t, A, \{\psi_k - \varphi_k\})A] \\
&\quad \cdot [\partial_t - b(t, A, \{\psi_k\})A + b(t, A, \{\psi_k - \varphi_k\})A] \\
&\quad - c(t, A, \{\psi_k\})A + c(t, A, \{\psi_k - \varphi_k\})A \\
&= \mathbf{P}(\{\psi_k\}) + a(t, A, \{\psi_k - \varphi_k\})A[\partial_t - b(t, A, \{\psi_k\})A] \\
&\quad + (\partial_t - a(t, A, \{\psi_k\})A + a(t, A, \{\psi_k - \varphi_k\})A)b(t, A, \{\psi_k - \varphi_k\})A \\
&\quad + c(t, A, \{\psi_k - \varphi_k\})A \\
&= \mathbf{P}(\{\psi_k\}) + R.
\end{aligned}$$

Now to apply Theorem 2.1 with $\mathbf{P}(\{\varphi_k\})$ and $\mathbf{P}(\{\psi_k\})$ instead of $\mathbf{P}_1$ and $\mathbf{P}_2$ or vice versa, we have to prove that R verifies property (2.12), and for this it is sufficient to prove that $a(t, A, \{\psi_k - \varphi_k\}), b(t, A, \{\psi_k - \varphi_k\}), c(t, A, \{\psi_k - \varphi_k\})$ verify property (2.12).

Let us take, for instance,

$$a(t, A, \{\chi_k\}) = \sum_1^\infty a_k(t)\chi_k(A)A^{-k},$$

where we have put $\chi_k = \psi_k - \varphi_k$.

Now it is sufficient to prove that there is an $\epsilon > 0$ such that for every $s \in \mathbf{R}$, $a(t, A, \{\chi_k\})$ is a holomorphic function on U with values in the Banach space $L(E_s, E_{s+\epsilon})$ of the bounded linear operators of E_s into $E_{s+\epsilon}$ and because for any given u for which $e^{sA}u$ is defined,

$$a(t, A, \{\chi_k\})e^{sA}u = e^{sA}a(t, A, \{\chi_k\})u,$$

it is enough to consider the case $s = 0$.

Because, by hypothesis, $a(t, A, \{\chi_k\})$ depends holomorphically on t when t varies in U, we have to prove that there is an $\epsilon > 0$, independent on t, such that for every $u \in H$, the series

$$\sum_0^\infty{}_h \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u$$

converges in H, i.e. that there is a constant M, independent of t and u, such that

$$\left\| \sum_0^\infty{}_h \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u \right\|_0 \leqslant M\|u\|_0. \tag{2.14}$$

We need the following

LEMMA 2.1. *There is a constant M_1, independent on $t \in U$, such that*

$$\|A^h a(t, A, \{\chi_k\})u\|_0 \leq M_1^{h+1} h! \|u\|_0 \tag{2.15}$$

for every nonnegative integer h and any $u \in H$.

PROOF. First of all we observe that $\chi_k(\lambda)$, for $\lambda \in \mathbf{R}$, is a real function, such that, $\forall k \geq 1$

$$0 \leq \chi_k(\lambda) \leq 1 \quad \text{and} \quad \operatorname{supp} \chi_k \subseteq [c_1 k, c_2 k + 1]$$

if we suppose $c_1 \leq c_2$.

Then by (2.4),

$$\|A^h a(t, A, \{\chi_k\})u\|_0 = \sum_1^{h-1} \sup_U |a_k(t)| \, \|A^{h-k}\chi_k(A)u\|_0$$

$$+ \sum_0^\infty \sup_U |a_{h+k}(t)| \, \|A^{-k}\chi_{k+h}(A)u\|_0$$

$$\leq \sum_1^{h-1} c^{k+1} k!(2c_2 k)^{h-k} \|u\|_0 + \sum_0^\infty c^{k+h+1}(k+h)! c_1^{-k}(h+k)^{-k} \|u\|_0$$

$$\leq \left\{(2c_2 h)^h \sum_0^\infty \left(\frac{c}{c_2}\right)^k + c^{h+1} h! \sum_0^\infty \left(\frac{c}{c_1}\right)^k\right\} \|u\|_0;$$

then if we take M_1 big enough and use the Stirling formula, we have (2.15).

Now we prove (2.14). For every integer l we have

$$\left\| \sum_{0}^{l}{}_h \frac{\epsilon^h}{h!} A^h a(t, A, \{\chi_k\})u \right\|_0 \leq \sum_0^l \frac{\epsilon^h}{h!} \|A^h a(t, A, \{\chi_k\})u\|_0$$

$$\leq \sum_{0}^{l}{}_h \frac{\epsilon^h}{h!} M_1^{h+1} h! \|u\|_0 \leq M_1 \sum_{0}^{l}{}_h (\epsilon M_1)^h \|u\|_0.$$

Then if $\epsilon < 1/M_1$, we have (2.14), with

$$M = M_1 \sum_{0}^{\infty}{}_h (\epsilon M_1)^h$$

which ends the proof of Theorem 2.2.

3. **Concatenations.** Starting from the "formal" operator **P** given by (2.1) and satisfying the hypotheses (2.1), (2.2) we are going to construct a sequence of formal operators $\mathbf{P}^0, \mathbf{P}^1, \ldots, \mathbf{P}^J, \ldots$ whose coefficients belong to $\mathcal{Q}_A((J))$, still satisfying the properties (2.1). For this construction, we refer to [1].

The first step is to find an operator $\psi(t, A) = \Sigma_k \psi_k(t) A^{-k} \in \mathcal{Q}_A((J))$, and a "coefficient" $c^0(A) = \Sigma_k c_k^0 A^{-k} \in C[[A^{-1}]]$, such that the operator

(3.1) $$\mathbf{P}^0 = (X + \psi(t, A))(Y - \psi(t, A)) - c^0(A)A$$

is formally equal to **P**.

Lemma II.4.1 in [1] shows how to construct $\psi(t, A)$ and $c^0(A)$.

Let us now put

$$X^0 = \partial_t - a^0(t, A)A, \qquad Y^0 = \partial_t - b^0(t, A)A,$$

$$a^0(t, A) = a(t, A) - \psi(t, A)A^{-1} = \sum_k a_k^0(t)A^{-k},$$

$$b^0(t, A) = b(t, A) + \psi(t, A)A^{-1} = \sum_k b_k^0(t)A^{-k}.$$

Then

(3.2) $$\mathbf{P}^0 = X^0 Y^0 - c^0(A)A.$$

REMARK 3.1. The important fact is that the "principal symbol" of **P** is the same as that of $\mathbf{P}^0$; in particular the leading coefficients in the power series $a(t, A)$ and $a^0(t, A)$ (respectively $b(t, A)$ and $b^0(t, A)$) are the same.

REMARK 3.2. If we suppose that property (2.4) is verified, then from the proof of Lemma II.4.1 in [1] it follows easily, after a shrinking of J, that there is a neighborhood V of 0 in C such that $\mathbf{R} \cap V = J$, on which $\psi_k(t)$ can be extended as a holomorphic function for every nonnegative integer k, we can suppose $V = U$.

Furthermore there is a constant $c_0 > 0$, such that

(3.3) $$\{|a_k^0(t)|, |b_k^0(t)|, |c_k^0|\} \leqslant c_0^{k+1}k!, \quad \forall k \in \mathbf{N} \text{ and } \forall t \in U.$$

By property (2.4), to prove (3.3) it is sufficient to see that there is a constant M and $D > 0$, such that

(3.4) $$\{|\psi_k(t)|, |c_k^0|\} \leqslant DM^k k!$$

$\forall k \in \mathbf{N}$, $\forall t \in U$ (possibly after a shrinking of U).

From (II.4.3) of [1] we have that $\psi_k(t)$ and c_k^0 have to verify the following equality

(3.5) $$\sum_{h=0}^{k} \delta_h(t)\psi_{k-h}(t) + c_k(t) - c_k^0 = \sum_{h=0}^{k-1} \psi_h(t)\psi_{k-1-h}(t) + \psi'_{k-1}(t)$$

where $\delta_h(t) = a_h(t) - b_h(t)$.

We reason by induction. We choose D so big that (3.4) is verified for $k = 0$, we suppose (3.4) true up to $k - 1$ and we want to prove it for k.

Because $\delta_0(0) = a_0(0) - b_0(0) = 0$, (3.5), computed for $t = 0$, gives us

(3.6) $$c_k^0 = c_k(0) + \sum_{h=1}^{k} \delta_h(0)\psi_{k-h}(0) - \sum_{h=1}^{k-1} \psi_h(0)\psi_{k-1-h}(0) + \psi'_{k-1}(0).$$

By (2.4), there is a constant c_1 such that $|\delta_k(t)|$, $|c_k(t)| \leqslant Dc_1^k k!$ if we suppose $D > 2c$. Then from (3.6) we have

$$|c_k^0| = \sum_{h=1}^{k} Dc_1^h h! \cdot DM^{k-h}(k-h)! + Dc_1^k k!$$
$$+ \sum_{1}^{k-1} D^2 M^{k-1} h!(k-h-1)! + \frac{DM^{k-1}}{d}(k-1)!.$$

Indeed, by Cauchy's inequality, if we shrink enough U, we have $|\psi'_{k-1}(t)| \leqslant (DM^{k-1}/d)(k-1)!$, $\forall t \in U$, where d is a positive constant independent on k.

Then if α is a constant such that $0 < \alpha \leqslant 1$,

$$|c_k^0| \leqslant \frac{\alpha}{6} DM^k k! \sum_{1}^{k} \left(\frac{6Dc_1}{\alpha M}\right)^h + \frac{\alpha}{6} D\left(\frac{6}{\alpha} c_1\right)^k k!$$
$$+ \frac{\alpha}{6} DM^k k! \left(\frac{6D}{\alpha M} \sum_{1}^{k-1} \frac{h!(k-1-h)!}{(k-1)!} \frac{1}{k} + \frac{6}{\alpha dkM}\right).$$

If we choose M so big that $M \geqslant 12Dc_1/\alpha$ and $M > (6D/\alpha + 6/d\alpha)$ the series $\Sigma_1^\infty (6Dc_1/\alpha M)^h$ converges. We have

(3.7) $$|c_k^0| \leqslant \tfrac{1}{2}\alpha DM^k k! \leqslant DM^k k!$$

which is the second inequality of (3.4).

From (3.5) we get

$$h_k(t) = \delta_0(t)\psi_k(t)$$
$$= c_k^0 - c_k(t) - \sum_{1}^{k}{}_h \delta_h(t)\psi_{k-h}(t) + \sum_{1}^{k-1}{}_h \psi_h(t)\psi_{k-1-h}(t) + \psi'_{k-1}(t)$$

for every $t \in U$.

As before, and from (3.7), we have for $\forall t \in U$

(3.8) $$|h_k(t)| \leqslant \tfrac{1}{2}\alpha DM^k k! + \tfrac{1}{2}\alpha DM^k k! \leqslant \alpha DM^k k!.$$

But $h_k(0) = 0$ and $\delta_0(0) = 0$; then

$$h_k(t) = \int_0^1 h'_k(\tau t)\, d\tau \cdot t, \qquad \delta_0(t) = \int_0^1 \delta'_0(\tau t)\, d\tau \cdot t.$$

By the Cauchy inequality and from (3.8), possibly after a shrinking of U, there is a positive constant d such that

$$|h_k'(t)| \leq \alpha(D/d)M^k k! \quad \forall k \in N.$$

Now, since $\delta_0'(0) \neq 0$, once more after shrinking U, we can suppose $\delta_0'(t) \neq 0$, $\forall t \in U$, then

$$\psi_h(t) = \int_0^1 h_k'(\tau t)\, d\tau \Big/ \int_0^1 \delta_0'(\tau t)\, d\tau, \qquad t \in U.$$

Let D_1 be equal to $|\int_0^1 \delta_0'(\tau t)\, d\tau|$, therefore

$$|\psi_k(t)| \leq (\alpha/dD_1)DM^k k!, \quad \forall t \in U.$$

Now if dD_1 is ≤ 1 we put $\alpha = dD_1$, otherwise we have $\alpha/dD_1 \leq 1$. In any case we get the first inequality of (3.4) which proves the result.

The operator $\mathbf{P}^0$ will be the first element in our sequence attached to $\mathbf{P}$. Suppose we have constructed the jth element $\mathbf{P}^j = X^jY^j - c^j(A)A$ we define

$$\begin{aligned} \mathbf{P}^{j+1} &= Y^jX^j - c^j(A)A = X^jY^j - [X^j, Y^j] - c^j(A)A \\ &= X^jY^j - \{c^j(A) + \delta_t^j(t, A)\}A \end{aligned} \tag{3.9}$$

where we put $\delta^j(t, A) = a^j(t, A) - b^j(t, A), j = 0, 1, \ldots$.

Then, as before, taking $\mathbf{P}^{j+1}$ instead of $\mathbf{P}$, we can write $\mathbf{P}^{j+1}$ in the way

$$\mathbf{P}^{j+1} = X^{j+1}Y^{j+1} - c^{j+1}(A)A \tag{3.10}$$

and we will have

$$X^{j+1} = \partial_t - a^{j+1}(t, A)A, \qquad Y^{j+1} = \partial_t - b^{j+1}(t, A)A,$$

$$\begin{aligned} a^{j+1}(t, A) &= a(t, A) - \psi^{j+1}(t, A)A^{-1}, \\ b^{j+1}(t, A) &= b(t, A) + \psi^{j+1}(t, A)A^{-1}, \end{aligned} \tag{3.11}$$

where $\psi^j(t, A) = \Sigma_k \psi_k^j(t)A^{-k} \in \mathcal{Q}_A((J)), c^j(A) = \Sigma_k c_k^j A^{-k} \in C[[A^{-1}]]$.

Remark 3.3. $\mathbf{P}^{j+1}$ will have the same "principal symbol" as $\mathbf{P}$, and therefore verifies (2.2) and (2.3). By Remark 3.2, like $\mathbf{P}^0$ it will verify a property analogous to (3.3), possibly after a shrinking of U, with $a_k^0(t)$, $b_k^0(t)$, c_k^0 and c_0 by $a_k^{j+1}(t)$, $b_k^{j+1}(t)$, c_k^{j+1}, c_{j+1}.

We will suppose that the constant c_j has the property

$$c_{j+1} > c_j > c, \qquad j \geq 0, \tag{3.12}$$

where c is the constant (2.4).

From (3.9) follows immediately the

Proposition 3.1. *We have, for* $j = 0, 1, \ldots,$

$$Y^j\mathbf{P}^j = \mathbf{P}^{j+1}Y^j, \tag{3.13}$$

$$X^j \mathbf{P}^{j+1} = \mathbf{P}^j X^j. \tag{3.14}$$

DEFINITION 3.1. The sequence of formal operators $\{P^j\}_{j\geq 0}$ will be called the *XY-concatenation* attached to $\mathbf{P}$.

PROPOSITION 3.2. *Let* $\{P^j\}$ *be the XY-concatenation attached to* $\mathbf{P}$. *Then the leading coefficient* c_0^j *of* $c^j(A)$ *is given by*

$$c_0^j = c_0(0) + j\delta_0'(0), \tag{3.15}$$

where $\delta_0(t)$ *is the leading coefficient in* $\delta^0(t, A)$.

See Proposition II.4.2 of [1].

THEOREM 3.1. *Let us suppose that* $\{\varphi_k\} \in [\widetilde{c}_0]$ *and* $\widetilde{c}_0 > 2c_0$, *where* c_0 *is given by* (3.3). *Then* $\mathbf{P}(\{\varphi_k\})$ *and* $\mathbf{P}^0(\{\varphi_k\})$ *are true operators and* $P(\{\varphi_k\})$ *is analytic hypoelliptic at* $t = 0$ *if and only if so is* $P^0(\{\varphi_k\})$.

PROOF. The proof is similar to that of Theorem 2.2. We observe that

$$\begin{aligned}
&A^{-1}(\mathbf{P}^0(\{\varphi_k\}) - \mathbf{P}(\{\varphi_k\}))\\
&\quad = \delta(t, A, \{\varphi_k\})\psi(t, A, \{\varphi_k\}) + c(t, A, \{\varphi_k\}) - c_0(A, \{\varphi_k\})\\
&\quad\quad - A^{-1}(\psi^2 + \psi_t)(t, A, \{\varphi_k\})\\
&\quad = \sum_0^\infty{}_k \left(\sum_0^k{}_h \delta_h(t)\psi_{k-h}(t)\varphi_h(A)\varphi_{k-h}(A)\right)A^{-k} + \sum_0^\infty{}_k (c_k(t) - c_k^0)\varphi_k(A)A^{-k}\\
&\quad\quad - \sum_0^\infty{}_k \left(\sum_0^k{}_h \psi_h(t)\psi_{k-h}\varphi_h(A)\varphi_{k-h}(A)\right)A^{-k-1} - \sum_0^\infty{}_k \psi_k'\varphi_k(A)A^{-k-1}\\
&\quad = \left\{\sum_0^\infty{}_k \left(\sum_0^k{}_h \delta_h(t)\psi_{k-h}(t)\right)\varphi_k(A)A^{-k} + \sum_0^\infty{}_k (c_k(t) - c_k^0)\varphi_k(A)A^{-k}\right.\\
&\quad\quad\quad \left. - \sum_0^\infty{}_k \left[\left(\sum_0^k{}_h \psi_h(t)\psi_{k-h}(t)\right) - \psi_k'(t)\right]\varphi_{k+1}(A)A^{-k-1}\right\}\\
&\quad\quad - \sum_0^\infty{}_k \sum_0^k{}_h \psi_{k-h}(t)\delta_h(t)(\varphi_k - \varphi_h\varphi_{k-h})(A)A^{-k}\\
&\quad\quad + \sum_0^\infty{}_k \sum_0^k{}_h \psi_h(t)\psi_{k-h}(t)(\varphi_{k+1} - \varphi_h\varphi_{k-h})(A)A^{-k-1}\\
&\quad\quad + \sum_0^\infty{}_k \psi_k'(t)(\varphi_{k+1} - \varphi_k)(A)A^{-k-1}.
\end{aligned}$$

For how $\mathbf{P}^0$ has been formally constructed, we have that the sum in the big brackets is zero, then

$$
\begin{aligned}
& A^{-1}(\mathbf{P}^0(\{\varphi_k\}) - \mathbf{P}(\{\varphi_k\})) \\
& \quad = -\sum_0^\infty{}_k \sum_0^k{}_h \delta_h(t)\psi_{k-h}(t)(\varphi_k - \varphi_h\varphi_{k-h})(A)A^{-k} \\
& \quad\quad + \sum_0^\infty{}_k \sum_0^k{}_h \psi_h(t)\psi_{k-h}(t)(\varphi_{k+1} - \varphi_h\varphi_{k-h})(A)A^{-k-1} \qquad (3.16)\\
& \quad\quad + \sum_0^\infty{}_k \psi_k'(t)(\varphi_{k+1} - \varphi_k)(A)A^{-k-1} \\
& \quad = \widetilde{R}.
\end{aligned}
$$

Then

$$\mathbf{P}^0(\{\varphi_k\}) - \mathbf{P}(\{\varphi_k\}) = A\widetilde{R} = R. \tag{3.17}$$

Now to apply Theorem 2.1, we-need to prove that R or $\widetilde{R}$ verifies property (2.12).

Then as in the proof of Theorem 2.2, we reduce ourselves to prove

LEMMA 3.1. *There is a constant M_1 independent on $t \in U$, such that*

$$\|A^h\widetilde{R}u\|_0 \leqslant M_1^{h+1}h!\,\|u\|_0 \tag{3.18}$$

for every nonnegative integer h and any $u \in H$.

PROOF. First of all we observe that: $\forall h,\ 0 \leqslant h \leqslant k,\ k \geqslant 1$

$$\varphi_h\varphi_{k-h}(\lambda) = \begin{cases} 0, & \lambda \leqslant \widetilde{c}_0 k/2, \\ 1, & \lambda \leqslant \widetilde{c}_0 k + 1. \end{cases}$$

Therefore

$$(\varphi_k - \varphi_h\varphi_{k-h})(\lambda) = \begin{cases} 0, & \lambda \leqslant \widetilde{c}_0 k/2,\ \lambda \geqslant \widetilde{c}_0 k + 1, \\ \leqslant 1, & \text{otherwise.} \end{cases}$$

$$(\varphi_{k+1} - \varphi_h\varphi_{k-h})(\lambda) = \begin{cases} 0, & \lambda \leqslant \widetilde{c}_0 k/2,\ \lambda \geqslant \widetilde{c}_0(k+1) + 1, \\ \leqslant 1, & \text{otherwise.} \end{cases} \tag{3.19}$$

$$(\varphi_{k+1} - \varphi_k)(\lambda) = \begin{cases} 0, & \lambda \leqslant \widetilde{c}_0 k,\ \lambda \geqslant \widetilde{c}_0(k+1) + 1, \\ \leqslant 1, & \text{otherwise.} \end{cases}$$

Now using the first of (3.19) we prove that (3.18) is verified from the sum

$$\sum_{0}^{\infty}{}_{k}\sum_{0}^{k}{}_{l}\,\delta_l(t)\psi_{k-l}(t)(\varphi_k-\varphi_l\varphi_{k-l})(A)A^{-k}. \tag{3.20}$$

We proceed in the same way for the other series in R. By (2.4), (3.3) and (3.12), $\forall t\in U$

$$\left\|A^h\sum_{0}^{\infty}{}_{k}\sum_{0}^{k}{}_{l}(\delta_l\psi_{k-l})(t)(\varphi_k-\varphi_l\varphi_{k-l})(A)A^{-k}u\right\|_0$$

$$\leqslant\sum_{1}^{h-1}{}_{k}\sum_{0}^{k}{}_{l}|(\delta_l\psi_{k-l})(t)|\,\|(\varphi_k-\varphi_l\varphi_{k-l})(A)A^{h-k}u\|_0$$

$$+\sum_{0}^{\infty}{}_{k}\sum_{0}^{k+h}{}_{l}|(\delta_l\psi_{k+h-l})(t)|\,\|(\varphi_{k+h}-\varphi_l\varphi_{k+h-l})(A)A^{-k}u\|_0$$

$$\leqslant\sum_{1}^{h-1}{}_{k}\sum_{0}^{k}{}_{l}4c_0^{k+2}l!(k-l)!(\widetilde{c}_0k+1)^{h-k}\cdot\|u\|_0$$

$$+\sum_{0}^{\infty}{}_{k}\sum_{0}^{k+h}{}_{l}4c_0^{k+h+2}l!(k+h-l)!\left[\frac{\widetilde{c}_0}{2}(k+h)\right]^{-k}\cdot\|u\|_0$$

$$\leqslant 4c_0^2\sum_{1}^{h-1}{}_{k}\,kc_0^k!(2\widetilde{c}_0k)^{h-k}\cdot\|u\|_0$$

$$+4c_0^2\sum_{0}^{\infty}{}_{k}(k+h)c_0^{k+h}(k+h)!\,\frac{2^k}{\widetilde{c}_0^{k}(k+h)^{-k}}\cdot\|u\|_0.$$

Then if M is a constant big enough we have

$$\leqslant M(2\widetilde{c}_0)^hh^h\sum_{0}^{\infty}{}_{k}\left(\frac{c_0}{\widetilde{c}_0}\right)^k\cdot\|u\|_0+Mc_0^hh!\sum_{0}^{\infty}{}_{k}\left(\frac{(2c_0)}{\widetilde{c}_0}\right)^k\cdot\|u\|_0.$$

Then because $\widetilde{c}_0>2c_0$, by Stirling formula, we have the (3.18).

With this lemma, we end the proof as in Theorem 2.2.

Let j be a positive integer and let $\mathbf{P}^0,\mathbf{P}^1,\ldots,\mathbf{P}^j$ be the first $j+1$ operators in the XY-concatenation attached to $\mathbf{P}$.

Let $\widetilde{c}_j$ be a constant bigger than $2c_j$, where c_j is the constant given by (3.12) and $\{\rho_k\}\in[\widetilde{c}_j]$.

Then all $\mathbf{P}^0(\{\rho_k\}),\mathbf{P}^1(\{\rho_k\}),\ldots,\mathbf{P}^j(\{\rho_k\})$ are true operators.

COROLLARY 3.1. *In the previous hypothesis, if for some integer* h, $0\leqslant h\leqslant j-1$, $\mathbf{P}^{h+1}(\{\rho_k\})$ *is analytic hypoelliptic at* $t=0$ *and we suppose that the following property is verified*:

there is an open neighborhood J' *of* 0, *relatively compact contained in* J, *such that if* $u \in \mathcal{D}'(J', H^{-\infty})$ *and*

$$Y^h(\{\rho_k\})u \in A(J', H^{\omega}), \qquad \mathbf{P}^h(\{\rho_k\})u \in A(J', H^{\omega}) \tag{3.21}$$

then $u \in A(J', H^{\omega})$

then also $\mathbf{P}^h(\{\rho_k\})$ *is analytic hypoelliptic at* $t = 0$.

PROOF. From (3.9), we have

$$\begin{aligned} &Y^h(\{\rho_k\})\mathbf{P}^h(\{\rho_k\}) \\ &\quad = [X^h(\{\rho_k\})Y^h(\{\rho_k\}) - \{c^h(A, \{\rho_k\}) + \delta_t^h(t, A, \{\rho_k\})\}A]\, Y^h(\{\rho_k\}) \\ &\quad = \widetilde{\mathbf{P}}^h(\{\rho_k\})Y^h(\{\rho_k\}) \end{aligned}$$

where we put $\widetilde{\mathbf{P}}^h = X^hY^h - \{c^h(A) + \delta_t^h(t, A)\}A$.

Now if we recall as $\mathbf{P}^{h+1}$ has been constructed, from Theorem 3.1 we have that $\widetilde{\mathbf{P}}^h(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$ if and only if so is $\mathbf{P}^{h+1}(\{\rho_k\})$.

Then by hypothesis, there is a neighborhood J'' of 0 that we can suppose equal to J', on which $\widetilde{\mathbf{P}}^h(\{\rho_k\})$ is hypoelliptic.

Then let $u \in \mathcal{D}'(J', H^{-\infty})$ such that $\mathbf{P}^h(\{\rho_k\})u \in A(J', H^{\omega})$, then

$$\widetilde{\mathbf{P}}^h(\{\rho_k\})Y^h(\{\rho_k\})u = Y^h(\{\rho_k\})\mathbf{P}^h(\{\rho_k\})u \in A(J', H^{\omega});$$

therefore $Y^h(\{\rho_k\})u \in A(J', H^{\omega})$ then by (3.21) $u \in A(J', H^{\omega})$, therefore $\mathbf{P}^h(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

4. Statement of the main theorem. Let $\mathbf{P}$ be the formal operator given by (2.1) and satisfying the hypothesis (2.2), (2.3) and (2.4) and let c be as usual, the constant given by (2.4).

We recall that for any sequence $\{\rho_k\} \in [c_1]$, where $c_1 > c$, $\mathbf{P}(\{\rho_k\})$ is a true operator.

DEFINITION 4.1. *The formal operator* $\mathbf{P}$ *is said to be analytic hypoelliptic at a point* $t \in J$ *if for any constant* $c_1 \geqslant c$ *and for any sequence* $\{\rho_k\} \in [c_1]$, *the true operator* $\mathbf{P}(\{\rho_k\})$ *is analytic hypoelliptic at* t.

REMARK 4.1. By Theorem 2.2, for $\mathbf{P}$ to be analytic hypoelliptic at a point $t \in J$, it is necessary and sufficient that there is a constant c_1 bigger than c and a sequence $\{\rho_k\} \in [c_1]$ for which $\mathbf{P}(\{\rho_k\})$ is analytic hypoelliptic at t.

We can now state the main theorem of this paper.

THEOREM 4.1. *Let* $\mathbf{P}$ *be the formal operator given by* (2.1) *and satisfying the hypotheses* (2.2), (2.3) *and* (2.4); *let* $\mathbf{P}^j = X^jY^j - c^j(A)A$, $j = 0, 1, \ldots,$ *denote the successive elements in the XY-concatenation attached to* $\mathbf{P}$.

The following are equivalent:

(4.1) $\mathbf{P}$ *is hypoelliptic at* $t = 0$;

(4.2) **P** *is analytic hypoelliptic at* $t = 0$;

(4.3) *For no integer* $j \geqslant 0$, *the formal power series in* A^{-1}, $c^j(A)$, *is identically zero.*

The equivalences (4.1) $\Longleftrightarrow$ (4.3) have been already proved in [1]. We have to prove that (4.3) implies (4.2) (by Definition 1.2, (4.2) $\Rightarrow$ (4.1)).

For the proof of Theorem 4.1 we need the following theorem which will be proved in §6.

THEOREM 4.2. *Let* I *be an open interval centered in the origin and* $\bar{I}$ *its closure. Suppose that* Q *is a (true) operator given by*

$$(4.4)\quad Q = (\partial_t - \alpha(t)tA - \widetilde{\alpha}(t, A))(\partial_t - \beta(t)tA - \widetilde{\beta}(t, A)) - \gamma(t)A - \widetilde{\gamma}(t, A)$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ *are complex valued* C^∞ *functions on* $\bar{I}$ *and* $\widetilde{\alpha}(t, A)$, $\widetilde{\beta}(t, A)$, $\widetilde{\gamma}(t, A)$ *are* C^∞ *functions on* $\bar{I}$ *with values in* $L(H, H)$, *for which the following hypotheses are verified*:

$$(4.5)\qquad \mathcal{Re}\,\alpha(0) > 0, \qquad \mathcal{Re}\,\beta(0) < 0;$$

$$(4.6)\qquad |\alpha(0) - \beta(0)|^2 \leqslant 2\mathcal{Re}\,\{\gamma(0)\overline{[\alpha(0) - \beta(0)]}\}.$$

Suppose furthermore that there is a constant $M > 0$, *such that for every* $t \in I$ *and any nonnegative integer* h

$$(4.7)\qquad \{|\partial_t^h\alpha(t)|, |\partial_t^h\beta(t)|, |\partial_t^h\gamma(t)|, \|\partial_t^h\widetilde{\alpha}(t, A)\|, \|\partial_t^h\widetilde{\beta}(t, A)\|, \|\partial_t^h\widetilde{\gamma}(t, A)\|\} \leqslant M^{h+1}h!$$

where $\|\ \|$ *is the norm in* $L(H, H)$.

Under these hypotheses **P** *is analytic hypoelliptic at* $t = 0$.

5. Proof of Theorem 4.1.

PROOF OF (4.3) $\Rightarrow$ (4.2). Let **P** be the operator given in Theorem 4.1 and j and integer such that

$$(5.1)\qquad j \geqslant \tfrac{1}{2} - \mathcal{Re}(c(0)/\delta_0'(0)).$$

Let $\widetilde{c}_j$ be a constant larger than $2c_j$, where c_j is given by (3.12), and $\{\rho_k\}$ a sequence belonging to $[\widetilde{c}_j]$; then $\mathbf{P}(\{\rho_k\})$, $\mathbf{P}^0(\{\rho_k\}), \ldots, \mathbf{P}^j(\{\rho_k\})$ are true operators.

First of all we observe that $\mathbf{P}^j(\{\rho_k\})$ verifies the hypotheses of Theorem 4.2.

Because the leading coefficients of $a^j(t, A)$, $b^j(t, A)$ are the same as those of $a(t, A)$, $b(t, A)$ by (2.2) and (2.3), we have that $\mathbf{P}^j(\{\rho_k\})$ can be written in the form (4.4); and (4.5) will be verified.

Furthermore by (3.15) and (5.1), since $\gamma(0) = c_0^j$, (4.6) is verified; finally by Remark 3.3 and by Cauchy's inequality used in the same way as in Remark

2.1, we see that also (4.7) is verified, therefore we may conclude that $\mathrm{P}^j(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

Now we want to prove that under the hypothesis (4.3), starting from the analytic hypoellipticity at $t = 0$ of $\mathrm{P}^j(\{\rho_k\})$, all the $\mathrm{P}^h(\{\rho_k\})$, $h = 0, 1, \ldots, j-1$, are analytic hypoelliptic at $t = 0$, which will imply by Theorem 3.1 that $\mathrm{P}(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$. By Remark 4.1 this will prove Theorem 4.1. We shall use Corollary 3.1, and we will have to prove that (3.21) is verified.

We will do the first step; for the others we proceed in the same way.

LEMMA 5.1. *Under the previous hypotheses (in particular, that* $\mathrm{P}^j(\{\rho_k\})$ *is hypoelliptic analytic in* J*), there is an open neighborhood* J' *of* 0, *relatively compact, contained in* J, *such that if* $u \in \mathcal{D}'(J', H^{-\infty})$ *and*

$$Y^{j-1}(\{\rho_k\})u \in A(J', H^{\omega}), \qquad \mathrm{P}^{j-1}(\{\rho_k\})u \in A(J', H^{\omega}), \tag{5.2}$$

then $u \in A(J', H^{\omega})$.

PROOF. We set $Y^{j-1}(\{\rho_k\})u = g$, $\mathrm{P}^{j-1}(\{\rho_k\})u = h$; then by (5.2), g and $h \in A(J', H^{\omega})$; therefore

$$c^{j-1}(A, \{\rho_k\})Au = X^{j-1}(\{\rho_k\})g - h \in A(J', H^{\omega}). \tag{5.3}$$

Let m_{j-1} be the smallest integer such that $c^{j-1}_{m_{j-1}} \neq 0$; such an integer exists, by (4.3).

Therefore we can find a number $\rho > 0$, so large that

$$\rho |c^{j-1}_{m_{j-1}}|^2 > \sup_{t \in j} |b_0(t)|. \tag{5.4}$$

Let

$$\begin{aligned}\gamma(A, \{\rho_k\}) &= \overline{c^{j-1}_{m_{j-1}}} \cdot c^{j-1}(A, \{\rho_k\})A^{m_{j-1}} \\ &= |c^{j-1}_{m_{j-1}}|^2 \rho_{m_{j-1}}(A) + \sum_{m_{j-1}+1}^{\infty} k\, \overline{c^{j-1}_{m_{j-1}}}\, c^{j-1}_k \rho_k(A) A^{m_{j-1}-k}.\end{aligned}$$

Then

$$\begin{aligned}\lambda(A, \{\rho_k\}) &= |c^{j-1}_{m_{j-1}}|^2 + \sum_{m_{j-1}+1}^{\infty} k\, \overline{c^{j-1}_{m_{j-1}}}\, c^{j-1}_k \rho_k(A) A^{m_{j-1}-k} \\ &= \gamma(A, \{\rho_k\}) + |c^{j-1}_{m_{j-1}}|^2 (1 - \rho_{m_{j-1}}(A)) = \gamma(A, \{\rho_k\}) + R,\end{aligned}$$

where $R = |c^{j-1}_{m_{j-1}}|^2(1 - \rho_{m_{j-1}}(A))$.

Now it is easy to see R verifies a property analogous to (3.18), then it will satisfy property (2.12) and by the fact that $\mathrm{P}^{j-1}(\{\rho_k\})u \in A(J', H^{\omega})$, we will get, as in the proof of Theorem 2.1, that $Ru \in A(J', H^\omega)$.

Then, by (5.3), $\rho\lambda(A, \{\rho_k\})Au \in A(J', H^\omega)$. Therefore

$$Z^{j-1}u = Y^{j-1}(\{\rho_k\})u - \rho\lambda(A, \{\rho_k\})Au \in A(J', H^\omega)$$

and by virtue of the ellipticity of Z^{j-1}, due to (5.4), we conclude that $u \in A(J', H^\omega)$, which proves the lemma.

As a consequence, by Corollary 3.1 we can conclude that $\mathbf{P}^{j-1}(\{\rho_k\})$ is analytic hypoelliptic at $t = 0$.

By repeating these arguments $j - 1$ times we reach the conclusion that $\mathbf{P}^0(\{\rho_k\})$ and (see Theorem 3.1) $\mathbf{P}(\{\rho_k\})$ are analytic hypoelliptic at $t = 0$.

This ends the proof of Theorem 4.1, by Remark 4.1.

6. **Proof of Theorem** 4.2. We follow closely the argument in [2]. Let us consider the operator

$$Q = (\partial_t - \alpha(t)tA - \widetilde{\alpha}(t, a))(\partial_t - \beta(t)tA - \widetilde{\beta}(t, A)) - \gamma(t)A - \widetilde{\gamma}(t, A).$$

If we redefine the coefficients appropriately, we can write Q in the form

$$Q = \partial_t^2 + \alpha(t)t\partial_t A + \beta(t)t^2A^2 + \gamma(t, A)\partial_t + \lambda(t, A)A + \mu(t, A) \tag{6.1}$$

where $\alpha(t)$, $\beta(t)$ are complex valued C^∞ functions and $\gamma(t, A)$, $\lambda(t, A)$, $\mu(t, A)$ are C^∞ functions, defined in I, with values in $L(H, H)$. There is a constant $M > 0$, such that, for every $t \in I$,

$$\{|\partial_t^h\alpha(t)|, |\partial_t^h\beta(t)|, \|\partial_t^h\gamma(t, A)\|, \|\partial_t^h\lambda(t, A)\|, \|\partial_t^h\mu(t, A)\|\} \leqslant M^{h+1}h! \tag{6.2}$$

where $\| \ \|$ is the norm in $L(H, H)$.

Because of the hypotheses (4.5) and (4.6), by Corollary II.2.1 of [1], if we shrink enough I, there is a constant $c > 0$, such that

$$\int(\|\partial_t v\|_0^2 + \|tAv\|_0^2)\,dt \leqslant c\left|\int(Qv, v)_0\,dt\right|, \quad \forall v \in C_c^\infty(I, H^\infty), \tag{6.3}$$

where $(\ ,\ )_0$ is the inner product of H.

If we use the norm $\| \ \|_{s,k}$ introduced in [4], the inequality (6.3) can be written

$$\|v\|_{1,1}^2 \leqslant c\left|\int(Qv, v)_0\,dt\right|,$$

whence, by Schwarz' inequality,

$$\|v\|_{1,1}^2 \leqslant c\|Qv\|_{-1,-1}\|v\|_{1,1},$$

which implies

$$\|v\|_{1,1} \leqslant c\|Qv\|_{-1,-1} \quad \forall v \in C_c^\infty(I, H^\infty). \tag{6.4}$$

Therefore, as shown in [4], (6.3) implies

$$\|v\|_{2,2} \leqslant c\|Qv\|_{0,0} = c\left\{\int \|Qv\|_0^2 \, dt\right\}^{1/2}, \tag{6.5}$$

$\forall v \in C_c^\infty(I, H^\infty)$. If we use the notation $\|f\|^2 = \int \|f\|_0^2 \, dt$, a norm equivalent to $\| \;\|_{2,2}$ is given by

$$N(v)^2 = \|\partial_t^2 v\|^2 + \|t\partial_t A v\|^2 + \|t^2 A^2 v\|^2 + \|\partial_t v\|^2 + \|Av\|^2 + \|v\|^2.$$

When the integration with respect to t is performed over some interval w we write $N_w(v)$.

Let us put

$$\begin{aligned} \Gamma_p &= \{(h, k) \in \mathbf{N}^2 \,|\, h \leqslant 4, h + k \leqslant p - h\} \\ &\cup \{(h, k) \in \mathbf{N}^2 \,|\, h \geqslant 4, h + k \leqslant p - 4\}. \end{aligned} \tag{6.6}$$

Note that Γ_p contains all the pairs (h, k) such that $h + k \leqslant p - 4$.

Now let:

$$N_\omega^p(v) = \sup_{(h,k) \in \Gamma_p} N_\omega(\partial_t^h A^k v) \tag{6.7}$$

and let $u \in \mathcal{D}'(I, H^{-\infty})$ be such that

$$Qu = f \in A(I, H^\omega). \tag{6.8}$$

We want to prove that there is an open subinterval J of I centered in 0, independent of u, such that

$$u \in A(J, H^\omega). \tag{6.9}$$

First of all (6.8) implies that

$$u \in C^\infty(I, H^\infty), \tag{6.10}$$

since Q is hypoelliptic in a suitable interval J (cf. [1, Corollary II.3.2])

Therefore, in order to prove (6.9) it suffices to show that we can choose J and a constant B such that

$$N_J^p(u) \leqslant B^{p+1} p! \quad \forall p \in N, p \geqslant p_0, \tag{6.11}$$

where p_0 is a fixed integer. By the standard embedding theorems, (6.11) will imply

$$\sup_J \|\partial_t^h A^k u\|_0 \leqslant C B_1^{h+k} (h + k)!$$

where B and C are suitable constants.

Possibly after shrinking I, we may suppose that, for some $c > 0$,

$$\sup_I \|\partial_t^h A^k Q u\|_0 \leqslant C^{h+k+1} (h + k)!. \tag{6.12}$$

Let $\omega =]-2, 2[$, we can always suppose that ω is contained in I and put $\omega_\delta =]-2 + \delta, 2 - \delta[$ with $0 \leqslant \delta < 2$.

Let ϵ and ϵ_1 be two real numbers, $0 < \epsilon, \epsilon_1 < 1$, and $\rho(t) \in C_c^\infty(\omega_{\epsilon_1})$ such that $\rho(t) = 1, \forall t \in \omega_{\epsilon+\epsilon_1}$, and $|\partial_t^\alpha \rho| \leqslant C_1 \epsilon^{-\alpha}, 0 \leqslant \alpha \leqslant 2$. By (6.5), if $p \geqslant \alpha$ we have that

$$N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) \leqslant N(\rho \partial_t^h A^k u) \leqslant C \| Q\rho \partial_t^h A^k u \|$$

$$\leqslant C\{\|\rho \partial_t^h A^k Q u\| + \|[Q, \rho]\partial_t^h A^k u\| + \|\rho[Q, \partial_t^h A^k] u\|\},$$

$$\forall (h, k) \in \Gamma_p.$$

From now on, C will denote a generic constant independent of h, k, p, ϵ and ϵ_1.

Now, by (6.12), we have

$$N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) \leqslant C\{C^p p! + \|[Q, \rho]\partial_t^h A^k u\| + \|\rho[Q, \partial_t^h A^k] u\|\}.$$

LEMMA 6.1. *Under the preceding hypotheses,*

$$\|[Q, \rho]\partial_t^h A^k u\| \leqslant C\{\epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u) + \epsilon^{-2} N_{\omega_{\epsilon_1}}^{p-2}(u)\}.$$

PROOF.

$$\|[Q, \rho]\partial_t^h A^k u\| \leqslant 2\|\rho' \partial_t \partial_t^h A^k u\| + \|\rho'' \partial_t^h A^k u\|$$

$$+ \|\alpha(t)\rho' t \partial_t^h A^{k+1} u\| + \|\gamma(t, A)\rho' \partial_t^h A^k u\|.$$

Let us estimate $\|\rho' \partial_t \partial_t^h A^k u\|$. Suppose $k \geqslant 1$; since:

(6.13) $$|\rho'/t| \leqslant C\epsilon^{-1},$$

we have

$$\|\rho' \partial_t \partial_t^h A^k u\| \leqslant C\epsilon^{-1} \|t \partial_t \partial_t^h A^k u\|_{\omega_{\epsilon_1}}$$

$$\leqslant C\epsilon^{-1} \|t \partial_t A(\partial_t^h A^{k-1} u)\|_{\omega_{\epsilon_1}} \leqslant C\epsilon^{-1} N_{\omega_{\epsilon_1}}(\partial_t^h A^{k-1} u).$$

If $h \geqslant 4, h + k \leqslant p - 4 \Rightarrow h + k - 1 \leqslant p - 1 - 4 \Rightarrow (h, k-1) \in \Gamma_{p-1}$; analogously, if $h \leqslant 4$, $(h, k-1) \in \Gamma_{p-1}$, then $\|\rho' \partial_t \partial_t^h A^k u\| \leqslant C\epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u)$.

Suppose $k = 0$. Then we can suppose $h \geqslant 1$ and

$$\|\rho' \partial_t \partial_t^h u\| \leqslant \|\rho' \partial_t \partial_t^{h-1} u\| \leqslant C\epsilon^{-1} N_{\omega_{\epsilon_1}}(\partial_t^{h-1} u) \leqslant C\epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u),$$

because, as before, we have $(h-1, 0) \in \Gamma_{p-1}$. Then in general we have $\|\rho' \partial_t \partial_t^h A^k u\| \leqslant C\epsilon^{-1} N_{\omega_{\epsilon_1}}^{p-1}(u)$.

Let us estimate $\|\alpha(t)\rho' t \partial_t^h A^{k+1} u\|$. Suppose $k \geqslant 1$; by (6.13) we have

$$\|\rho' t\partial_t^h A^{k+1}u\| \leqslant C\epsilon^{-1}\|t^2A^2(\partial_t^h A^{k+1}u)\|_{\omega_{\epsilon_1}}$$

$$\leqslant C\epsilon^{-1}N_{\omega_{\epsilon_1}}(\partial_t^h A^{k-1}u) \leqslant C\epsilon^{-1}N_{\omega_{\epsilon_1}}^{p-1}(u).$$

If $k = 0$ we can suppose $h \geqslant 1$; then

$$\|\rho' t\partial_t^h Au\| \leqslant C\epsilon^{-1}\|t\partial_t A(\partial_t^{h-1}u)\|_{\omega_{\epsilon_1}}$$

$$\leqslant C\epsilon^{-1}N_{\omega_{\epsilon_1}}(\partial_t^{h-1}u) \leqslant C\epsilon^{-1}N_{\omega_{\epsilon_1}}^{p-1}(u).$$

Then in general $\|\alpha(t)\rho' t\partial_t^h A^{k+1}u\| \leqslant C\epsilon^{-1}N_{\omega_{\epsilon_1}}^{p-1}(u)$.

If we consider $\|\rho''\partial_t^h A^k u\|$, $\|\gamma(t, A)\rho'\partial_t^h A^k u\|$, we have that both of them are $\leqslant C\epsilon^{-2}\|\partial_t^h A^k u\|_{\omega_{\epsilon_1}}$.

Now if $h \geqslant 2$,

$$\|\partial_t^h A^k u\|_{\omega_{\epsilon_1}} = \|\partial_t^2(\partial_t^{h-2}A^k u)\|_{\omega_{\epsilon_1}} \leqslant N_{\omega_{\epsilon_1}}(\partial_t^{h-2}A^k u) \leqslant N_{\omega_{\epsilon_1}}^{p-2}(u);$$

if $h = 1$,

$$\|\partial_t A^k u\|_{\omega_{\epsilon_1}} \leqslant N_{\omega_{\epsilon_1}}(A^k u) \leqslant N_{\omega_{\epsilon_1}}^{p-2}(u),$$

because $1 + k \leqslant p - 1 \Rightarrow k \leqslant p - 2 \Rightarrow (0, k) \in \Gamma_{p-2}$.

If $h = 0$, we can suppose $k \geqslant 2$; then, since $|\rho''/t^2| \leqslant c\epsilon^{-2}$, and by (6.13),

$$\|\rho'' A^k u\|,\ \|\gamma(t, A)\rho' A^k u\| \leqslant C\epsilon^{-2}\|t^2 A^k u\|_{\omega_{\epsilon_1}}$$

$$\leqslant c\epsilon^{-2}N_{\omega_{\epsilon_1}}(A^{k-2}u) \leqslant c\epsilon^{-2}N_{\omega_{\epsilon_1}}^{p-2}(u)$$

then in general

$$\|\rho''\partial_t^h A^k u\|,\ \|\gamma(t, A)\rho'\partial_t^h A^k u\| \leqslant c\epsilon^{-2}N_{\omega_{\epsilon_1}}^{p-2}(u),$$

which proves the lemma.

LEMMA 6.2. *Same hypotheses as in Lemma* 6.1.

$$\|\rho[Q, \partial_t^h A^k]u\| \leqslant C\sum_{k=1}^{p} C^k \frac{p!}{(p-k)!}N_{\omega_{\epsilon_1}}^{p-k}(u). \tag{6.14}$$

PROOF.

$$\|\rho[Q, \partial_t^h A^k]u\| = \|\rho[Q, \partial_t^h]A^k u\|$$

$$\begin{aligned}\leqslant\ & \|\rho[\alpha(t)t, \partial_t^h]\partial_t A^{k+1}u\| \\ & + \|\rho[\beta(t)t^2\partial_t^h]A^{k+2}u\| + \|\rho[\gamma(t, A), \partial_t^h]\partial_t A^k u\| \\ & + \|\rho[\lambda(t, A), \partial_t^h]A^{k+1}u\| + \|\rho[\mu(t, A), \partial_t^h]A^k u\|.\end{aligned} \tag{6.15}$$

We may suppose $h \geqslant 1$. We have, first,

$$\|\rho[\alpha(t)t, \partial_t^h]\partial_t A^{k+1}u\|$$

$$\leqslant \sum_{\lambda+\nu=h;\nu<\alpha} \frac{h!}{\lambda!\nu!} \|(\partial_t^\lambda \alpha)\rho t\partial_t A(\partial_t^\nu A^k u)\|$$

$$+ \sum_{\lambda+\nu+1=h} \frac{h!}{\lambda!\nu!} \|(\partial_t^\lambda \alpha)\rho(\partial_t^{\nu+1} A^{k+1} u)\|.$$

By (6.2), it suffices to prove that

$$\|\rho t\partial_t A(\partial_t A^k u\| \leqslant CN^{p-\lambda}_{\omega_{\epsilon_1}}(u),$$

(6.16)
$$\|\rho\partial_t^{\nu+1} A^{k+1} u\| \leqslant CN^{p-\lambda-1}_{\omega_{\epsilon_1}}(u).$$

Suppose $h \geqslant 4$. For the first inequality (6.16), with $(h, k) \in \Gamma_p$, we have $\lambda + \nu + k \leqslant p - 4 \Rightarrow p - \lambda - 4 \Rightarrow (\nu, k) \in \Gamma_{p-\lambda}$. Consequently,

$$\|\rho t\partial_t A(\partial_t^\nu A^k u)\| \leqslant CN_{\omega_{\epsilon_1}}(\partial_t^\nu A^k u) \leqslant CN^{p-\lambda}_{\omega_{\epsilon_1}}(u).$$

As for the second inequality (6.16) we have (1) if $\nu \geqslant 1, \lambda + \nu + 1 + k \leqslant p - 4 \Rightarrow \nu - 1 + k + 1 \leqslant (p - \lambda - 1) - 4 \Rightarrow (\nu - 1, k + 1) \in \Gamma_{p-\lambda-1}$, and therefore

$$\|\rho(\partial_t^{\nu+1} A^{k+1} u)\| \leqslant CN_{\omega_{\epsilon_1}}(\partial_t^{\nu-1} A^{k+1} u) \leqslant CN^{p-\lambda-1}_{\omega_{\epsilon_1}}(u);$$

(2) if $\nu = 0, \lambda + 1 + k \leqslant p - 4 \Rightarrow 1 + k \leqslant p - \lambda - 4 \leqslant (p - \lambda - 1) - 4 \Rightarrow (0, k + 1) \in \Gamma_{p-\lambda-1}$, and

$$\|\rho\partial_t A^{k+1} u\| \leqslant CN_{\omega_{\epsilon_1}}(A^{k+1} u) \leqslant CN^{p-\lambda-1}_{\omega_{\epsilon_1}}(u).$$

Similar argument when $h \leqslant 4$. As a consequence of (6.16) we see that $\|\rho[\alpha(t)t, \partial_t^h]\partial_t Au\|$ verifies the estimate (6.14).

We operate in the same way with $\|\rho[\beta(t)t^2, \partial_t^h]A^{k+2}u\|$.

Next we see, by (6.2), that

$$\|\rho[\gamma(t, A), \partial_t^h]\partial_t A^k u\| \leqslant \sum_{\nu<h} \binom{h}{\nu} \|\rho\partial_t^{h-\nu}\gamma(t, A) \cdot \partial_t^{\nu+1} A^k u\|$$

$$\leqslant \sum_{\nu<h} \binom{h}{\nu} c^{h-\nu+1}(h-\nu)!\|\rho\partial_t(\partial_t^\nu A^k u)\|$$

$$\leqslant \sum_{\nu<h} c^{h-\nu+1} \frac{h!}{\nu!} N_{\omega_{\epsilon_1}}(\partial_t^\nu A^k u).$$

But if $h \geqslant 4, h + k \leqslant p - 4 \Rightarrow \nu + k \leqslant p - (h - \nu) - 4 \Rightarrow (\nu, k) \in \Gamma_{p-(h-\nu)}$. If $h \leqslant 4, h + k \leqslant p - h \Rightarrow \nu + k \leqslant p - (h - \nu) - h \leqslant p - (h - \nu) - \nu \Rightarrow$

$(\nu, k) \in \Gamma_{p-(h-\nu)}$. Therefore

$$\|\rho[\gamma(t, A), \partial_t^h]\partial_t A^k u\| \leqslant \sum_{\nu<h} c^{h-\nu+1} \frac{h!}{\nu!} N_{\omega_{\epsilon_1}}^{p-(h-\nu)}(u).$$

We obtain a similar estimate for $\|\rho[\lambda(t, A), \partial_t^h]A^{k-1}u\|$ and $\|\rho[\mu(t, A), \partial_t^h]A^k u\|$ and the lemma is thus proved.

By Lemmas 6.1 and 6.2, we get that there is a constant $c > 0$ such that, $\forall p \in N, p \geqslant 4, \forall (h, k) \in \Gamma_p$,

$$N_{\omega_{\epsilon+\epsilon_1}}(\partial_t^h A^k u) \leqslant C\left\{C^p p! + \sum_{k=1}^{p} C^k \frac{p!}{(p-k)!} N_{\omega_{\epsilon_1}}^{p-k}(u) + \sum_1^2 {}_k\epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u)\right\},$$

whence:

$$N_{\omega_{\epsilon+\epsilon_1}}^p(u) \leqslant C\left\{C^p p! + \sum_{k=1}^{p} C^k \frac{\epsilon^k p!}{(p-k)!} \epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u) + \sum_{k=1}^{2} \epsilon^{-k} N_{\omega_{\epsilon_1}}^{p-k}(u)\right\}. \tag{6.17}$$

Let l be an integer such that $l \geqslant p + 4$, $\epsilon = 1/l$, $\epsilon_1 = p/l$ and put

$$d_p = N_{\omega_{(p+1)/l}}^p(u) l^{-p-4}.$$

The inequality (6.17) can be rewritten

$$d_p \leqslant C\left\{C^p + \sum_{k=1}^{p} C^k d_{p-k} + \sum_{k=1}^{2} d_{p-k}\right\}$$

for every integer $p \geqslant 4$, $p \leqslant l - 4$, which easily implies, for some $B > 0$,

$$d_p \leqslant B^{p+1} \quad \forall p \leqslant l - 4. \tag{6.18}$$

In turn (6.18) implies, for all $l \geqslant 4$,

$$N_{\omega_1}^{l-4}(u) \leqslant B^{l-3} l^l.$$

Finally, after an increase of the constant B we get (6.11) with $J = \omega_1$.

REFERENCES

1. F. Trèves, *Concatenations of second order evolution equation applied to local solvability and hypoellipticity*, Comm. Pure Appl. Math. 26 (1973), 201–250.

2. V. V. Grušin, *On a class of elliptic pseudodifferential operators that are degenerate*

on a submanifold, Mat. Sb. 84 (126) (1971), 163–195 = Math USSR Sb. 13 (1971), 155–186. MR 44 #860.

3. F. Trèves, *Ovcyannicov theorem and hyperdifferential operators*, Notas de Matemática, no. 46, Instituto de Matematica Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro, 1968. MR 44 #7386.

4. L. Boutet De Monvel and F. Trèves, *On a class of pseudodifferential operators with double characteristics*, Invent. Math. 24 (1974), 1–34.

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